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THOMSON REUTERS

[FRTSSDS- June 2018]
DOI: 10.5281/zenodo.1296234

ISSN 2348 – 8034
Impact Factor- 5.070

GLOBAL JOURNAL OF ENGINEERING SCIENCE AND RESEARCHES

APPLICATION OF FIXED POINT THEOREMS FOR THREE COMMUTING MAPPINGS IN PM – SPACE

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ABSTRACT

Various revolutionary applications of fixed point theorems are generalized by presenting hypothesis of functions. A series of articles have been made in latest two decades that recommend generalizations and extensions of the Banach Contraction Principle, The Principle states about a contraction f of a complete metric space (X,d) has an exclusive fixed point.

Key words: *Fixed points, mappings, Commuting, PM - Space*

I. INTRODUCTION

Typical approaches have been either to vary the contraction requirement that d(fx, fy) < r d(x, y) for some r 6 (0,1) and all x,y 6 X, or to introduce more functions with conditions appended. For example, in 1976 the following result appeared Jungek generalized the Banach contraction principle by introducing a contraction condition for a pair of commuting mappings on metric spaces. Further, Jungek's result has been extended on several settings.

The concept of weak commutative maps was studied by Seesa and generalized the result of Das and Naik by considering a pair of self maps P,T of a metric space (X,d) satisfying a weaker condition than commutative ; namely

$$d\,(PTx,\ TPx)\ \leq d\,(Tx,\ Px)$$

For each $x \in X$.

Employing an idea of Fisher, Rhoades established some common fixed point theorems and generalized results of Das and Naik , Fisher and Sessa.

Tiwari and Pant proved a common fixed point theorem for a pair of commuting mappings in PM space and some fixed point theorems for triplet of mappings have been investigated by Singh and Pant .

In this section we shall prove a common fixed point theorem for three commuting mappings in PM space by using the concept of Rhoads andTiwari and Pant .

Definition 1 : Two self mappings g and h of a PM space (X, F, t) are weakly commutative iff

$$F\,(ghu,\ hgu\ ;\ x)\ \geq\ F\,(hu,\ gu\ ;\ x)\quad \text{for all}\ \ u \in X \text{ and } x > 0.$$

Definition 2 : A sequence $\{u_n\}$ in X is said to converge to a point p in X iff for every $\varepsilon > 0$ and $\lambda > 0$ there is an integer M (ε, λ) such that

$$F_{u_n p}\,(\varepsilon) > 1 - \lambda$$

for all $n \geq M(\varepsilon, \lambda)$.

Further the sequence $\{u_n\}$ is called Cauchy sequence iff for every $\varepsilon > 0$ and $\lambda > 0$ there is an integer M (ε, λ) such that

$$F_{u_n, u_m}\,(\varepsilon) > 1 - \lambda$$

for all $n,m \geq M(\varepsilon, \lambda)$.

Definition 3 [61] : A sequence of mappings $T_n : X \to X$ on PM space X converges uniformly to a mapping $T : X \to X$ iff for every $\varepsilon > 0$ and $\lambda > 0$ there exists a positive integer $M = M(\varepsilon, \lambda)$ such that

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$$F_{Tp,T_n p}(\varepsilon) > 1-\lambda$$

for every $p \in X$ and all $n \geq M$.

Definition 4 [61] : A sequence of mappings $T_n : X \to X$ on a PM space X converges pointwise to a mapping $T : X \to X$ iff for every $u \in X$, $\{T_n(u)\}$ converge to $T(u)$.

Definition 5 [142] : Three mappings P, S and T on a PM space (X,) will be called a generalized contraction triplet (P, S ; T) iff there exists $h \in (0,1)$ such that for every $u, v \in X$

$$F_{pu,Sv}(hx) \geq \min\{F_{Tu,Tv}(x), F_{Pu,Tu}(x), F_{Sv,Tv}(x), F_{Pu,Tv}(2x), F_{Sv,Tu}(2x)\}$$

holds for all $x > 0$.

Definition 6 [143] : Three mappings P, S and T on a PM space (X,) will be called a generalized contraction triplet (P, S ; T) iff there exists $h \in (0,1)$ such that for every $u, v \in X$

$$F_{Pu,Pv}(hx) \geq \min\{F_{Su,Tv}(x), F_{Pu,Su}(x), F_{Pv,Tv}(x), F_{Pu,Tv}(2x), F_{Pv,Su}(2x)\}$$

forall $x > 0$.

Definition 7 : Four mappings P, Q, S and T on a PM space (X, F, t) will be called a generalized contraction quadruplet (P, Q ; S, T) iff there exists $h \in (0,1)$ such that for every $u, v \in X$

$$F_{Pu,Qv}(hx) \geq \min\{F_{Su,Tv}(x), F_{Pu,Su}(x), F_{Tv,Qv}(x), F_{Su,Qv}(2x), F_{Pu,Tv}(2x)\}$$

forall $x > 0$.

Definition 8 : A Menger space is a triplet (X, F, t) consisting of a probabilistic metric space (X, F) and a t-norm satisfying the inequality,

$$F_{p,r}(x+y) > t\{F_{p,q}(x), F_{q,r}(y)\} \qquad \text{-------- (1)}$$

forall $p, q, r \in X$ and $x \geq 0$, $y \geq 0$.

We shall use the following lemma to prove our main result.

Lemma 9 : Let $\{u_n\}$ be a sequence to complete Menger space (X, F, t) where t is continuous and satisfying $t(x, x) \geq x$ for $x \in [0,1]$. If there exists $q \in [0,1]$ such that

$$F_{u_n,u_{n+1}}(qx) \geq F_{u_{n-1},u_n}(x) \quad \forall \quad n = 1,2,3 \ldots \ldots \ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

forall $x > 0$, then $\{u_n\}$ converges to a fixed point u in X.

Theorem 10 : Let (X, F, t) be a complete Menger space where t is continuous and satisfies $t(x,x) \geq x$, for $x \in [0,1]$. Let P, S and T be commuting mappings from X into itself such that

$$\left.\begin{array}{l} P(X) \subseteq T X) \\ \text{and} \quad S(X) \subseteq T(X) \end{array}\right\} \qquad \text{-------- (3)}$$

$$F_{Pu,Pv}(qx) \geq h \min\{F_{Tu,Sv}(x), F_{Su,Tv}(x), F_{Tu,Pu}(x), F_{Tv,Pv}(x), F_{Su,Pu}(x), F_{Sv,Pv}(x), F_{Tu,Pv}(x), F_{Tv,Pu}(x), F_{Su,Pv}(x), F_{Sv,Pu}(x)\} \qquad \text{-------- (4)}$$

where $0 < h < 1$

and for all $u, v \in X$ and for all $x > 0$, where $q \in (0,1)$.

If T be continuous, then P, S and T have a unique common fixed point.

Proof : Pick $x_0 \in X$. In view of (3), we can construct a sequence $\{u_n\}$ in X such that

$$\left.\begin{array}{l} Tu_n = Pu_{n-1} \ \text{ or } \ Tu_{n+1} = Pu_n \\ \text{and} \quad Su_{n-1} = Tu_n \ \text{ or } \ Su_n = Tu_{n+1} \end{array}\right\} \qquad \text{-------- (5)}$$

for all $n = 1,2,3 \ldots \ldots$

Now by (4),

$$F_{Tu_n,Tu_{n+1}}(qx) = F_{Pu_{n-1},Pu_n}(qx)$$

[FRTSSDS- June 2018] ISSN 2348 – 8034
DOI: 10.5281/zenodo.1296234 Impact Factor- 5.070

$\geq h \min \{ F_{Tu_{n-1},Su_n}(x), F_{Su_{n-1},Tu_n}(x), F_{Tu_{n-1},Pu_{n-1}}(x), F_{Tu_n,Pu_n}(x),$
$F_{Su_{n-1},Pu_{n-1}}(x), F_{Su_n,Pu_n}(x), F_{Tu_{n-1},Pu_n}(x), F_{Tu_n,Pu_{n-1}}(x),$
$F_{Su_{n-1},Pu_n}(x), F_{Su_n,Pu_{n-1}}(x) \}$
$\geq h \min \{ F_{Tu_{n-1},Tu_{n+1}}(x), F_{Tu_n,Tu_n}(x), F_{Tu_{n-1},Tu_n}(x), F_{Tu_n,Tu_{n+1}}(x),$
$F_{Tu_n,Tu_n}(x), F_{Tu_{n+1},Tu_{n+1}}(x), F_{Tu_{n-1},Tu_{n+1}}(x), F_{Tu_n,Tu_n}(x),$
$F_{Tu_n,Tu_{n+1}}(x), F_{Tu_{n+1},Tu_n}(x) \}$
$\geq h \min \{ F_{Tu_{n-1},Tu_{n+1}}(x), F_{Tu_n,Tu_n}(x), F_{Tu_{n-1},Tu_n}(x), F_{Tu_{n+1},Tu_{n+1}}(x),$
$F_{Tu_n,Tu_{n+1}}(x) \}$
$\geq h F_{Tu_{n-1},Tu_n}(x)$

So $F_{Tu_n,Tu_{n+1}}(qx) \geq h F_{Tu_{n-1},Tu_n}(x)$
Similarly $F_{Tu_n,Tu_{n+1}}(qx) = F_{Su_{n-1},Su_n}(qx)$
=> $F_{Tu_n,Tu_{n+1}}(qx) \geq h F_{Tu_{n-1},Tu_n}(x)$

By (5), $\{Pu_n\}$, $\{Su_n\}$ also converges to u, further

$F_{PTu_n,PTu_{n+1}}(qx) = F_{PPu_{n-1},PPu_n}(qx)$
$\geq h \min \{ F_{PTu_{n-1},PSu_n}(x), F_{PSu_{n-1},PTu_n}(x),$
$F_{PTu_{n-1},PPu_{n-1}}(x), F_{PTu_n,PPu_n}(x),$
$F_{PSu_{n-1},PPu_{n-1}}(x), F_{PSu_n,PPu_n}(x),$
$F_{PTu_{n-1},PPu_n}(x), F_{PTu_n,PPu_{n-1}}(x),$
$F_{PSu_{n-1},PPu_n}(x), F_{PSu_n,PPu_{n-1}}(x) \}$

$\geq h \min \{ F_{PTu_{n-1},PTu_{n+1}}(x), F_{PTu_n,PTu_n}(x),$
$F_{PTu_{n-1},PTu_n}(x), F_{PTu_n,PTu_{n+1}}(x),$
$F_{PTu_n,PTu_n}(x), F_{PTu_{n+1},PTu_{n+1}}(x),$
$F_{PTu_{n-1},PTu_{n+1}}(x), F_{PTu_n,PTu_n}(x),$
$F_{PTu_{n-1},PTu_{n+1}}(x), F_{PTu_{n+1},PTu_n}(x) \}$
$\geq h \min \{ F_{PTu_{n-1},PTu_{n+1}}(x), F_{PTu_n,PTu_n}(x),$
$F_{PTu_{n-1},PTu_n}(x), F_{PTu_{n+1},PTu_{n+1}}(x),$
$F_{PTu_n,PTu_{n+1}}(x) \}$
$\geq h F_{PTu_{n-1},PTu_n}(x)$

So $F_{PTu_n,PTu_{n+1}}(qx) \geq h F_{PTu_{n-1},PTu_n}(x)$
Similarly $F_{STu_n,STu_{n+1}}(qx) = F_{SSu_{n-1},SSu_n}(qx)$
=> $F_{STu_n,ST_{n+1}}(qx) \geq h F_{STu_{n-1},STu_n}(x)$

Thus $\{PTu_n\}$ and $\{STu_n\}$ are also fundamental in X.
Consequently,

$$\lim_n F_{PTu_n,\ TTu_n}(x) = \lim_n F_{PTu_n,PTu_{n-1}}(x) = 1$$

and

$$\lim_n F_{STu_n,\ TTu_n}(x) = \lim_n F_{STu_n,STu_{n-1}}(x) = 1$$

By the continuity of T,

$PTu_n = TPu_n = TTu_{n+1} \rightarrow T_u$
and $STu_n = TSu_n = TTu_{n+1} \rightarrow T_u$

Further,

$F_{PTu_n,Pu}(qx) \geq h \min \{ F_{TTu_n,Su}(x), F_{STu_n,Tu}(x),$
$F_{TTu_n,PTu_n}(x), F_{Tu,Pu}(x),$
$F_{STu_n,PTu_n}(x), F_{Su,Pu}(x),$
$F_{TTu_n,Pu}(x), F_{Tu,PTu_n}(x),$
$F_{STu_n,Pu}(x), F_{Su,PTu_n}(x) \}$

[FRTSSDS- June 2018]
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ISSN 2348 – 8034
Impact Factor- 5.070

$$\geq h \min \{ F_{TTu_n,Su}(x), F_{TSu_n,Tu}(x), F_{TTu_n,TPu_n}(x), F_{Tu,Pu}(x), F_{Tu\ ,Tu}(x), F_{Su,Pu}(x), F_{TTu_n,Pu}(x), F_{Tu,TPu_n}(x), F_{Tu\ ,Pu}(x), F_{Su,Tu}(x) \}$$

$$\geq h \min \{ F_{Tu,Su}(x), F_{Tu,Tu}(x), F_{Tu,Tu}(x), F_{Tu,Pu}(x), F_{Tu\ ,Tu}(x), F_{Su,Pu}(x), F_{Tu,Pu}(x), F_{Tu,Tu}(x), F_{Tu\ ,Pu}(x), F_{Su,Tu}(x) \}$$

$$\geq h \min \{ F_{Tu,Pu}(x), F_{Tu,Tu}(x), F_{Tu,Tu}(x), F_{Tu,Pu}(x), F_{Tu\ ,Tu}(x), F_{Pu,Pu}(x), F_{Tu,Pu}(x), F_{Tu,Tu}(x), F_{Tu\ ,Pu}(x), F_{Pu,Tu}(x) \}$$

$$\geq h F_{Tu\ ,Pu}(x)$$

So $\quad F_{PTu_n,Pu}(qx) \geq h\ F_{Tu\ ,Pu}(x)$

whence $\quad Tu = Pu$ -------- (6)

Similarly $\quad F_{STu_n,Su}(qx) \geq h\ F_{Tu\ ,Su}(x)$

whence $\quad Tu = Su$ -------- (7)

Consequently,

$$\left.\begin{array}{l} T(Tu) = T(Pu) = P(Tu) = P(Pu) \\ \text{and} \quad T(Tu) = T(Su) = S(Tu) = S(Su) \end{array}\right\} \quad \text{-------- (8)}$$

By (4)

$$F_{Pu,P(Pu)}(qx) \geq h \min \{ F_{Tu,S(Pu)}(x), F_{Su,T(Pu)}(x), F_{Tu\ ,Pu}(x), F_{T(Pu),P(Pu)}(x), F_{Su,Pu}(x), F_{S(Pu),P(Pu)}(x), F_{Tu,P(Pu)}(x), F_{T(Pu),Pu}(x), F_{Su,P(Pu)}(x), F_{S(Pu),Pu}(x) \}$$

By using (6), (7) and (8),l we get

$$\geq h \min \{ F_{Tu,P(Pu)}(x), F_{Pu,P(Pu)}(x), F_{Tu\ ,Pu}(x), F_{P(Pu),P(Pu)}(x), F_{Pu,Pu}(x), F_{P(Pu),P(Pu)}(x), F_{Tu,P(Pu)}(x), F_{T(Pu),Pu}(x), F_{Pu,P(Pu)}(x), F_{P(Pu),Pu}(x) \}$$

$$\geq h F_{Pu,P(Pu)}(x)$$

So $\quad F_{Pu,P(Pu)}(qx) \geq h\ F_{Pu,P(Pu)}(x)$

whence $\quad P(Pu) = Pu$

Thus Pu is a common fixed point of P, S and T.

To prove the uniqueness of common fixed point of P, S and T, let P, S and T have two common fixed point ξ and η then,

$$F_{\xi,\eta}(qx) = F_{P\xi,P\eta}(qx)$$
$$\geq h \min \{ F_{T\xi,S\eta}(x), F_{S\xi,T\eta}(x), F_{T\xi,P\xi}(x), F_{T\eta,P\eta}(x), F_{S\xi,P\xi}(x), F_{T\eta,P\eta}(x), F_{T\xi,P\eta}(x), F_{T\eta,P\xi}(x), F_{S\xi,P\eta}(x), F_{S\eta,P\xi}(x) \}$$

[FRTSSDS- June 2018] ISSN 2348 – 8034
DOI: 10.5281/zenodo.1296234 Impact Factor- 5.070

$\geq h \min \{ F_{\xi,\eta}(x), F_{\xi,\eta}(x), \quad F_{\xi,\xi}(x), F_{\eta,\eta}(x),$
$F_{\xi,\xi}(x), F_{\eta,\eta}(x), \quad F_{\xi,\eta}(x), F_{\eta,\xi}(x),$
$F_{\xi,\eta}(x), F_{\eta,\xi}(x) \}$
$\geq h F_{\xi,\eta}(x)$

So $F_{\xi,\eta}(qx) \geq h \; F_{\xi,\eta}(x)$

proving $\xi = \eta$.

II. CONCLUSION

Thus P, S and T have a unique common fixed point.The concept of PM-spaces may have very important applications in quantum particle physics particularly in connections with both string and ∞ e theory, which were introduced and studied by a well-known scientist, Mohamed Saladin El Naschie. It is also of fundamental importance in probabilistic functional analysis, nonlinear analysis and applications. In the theory of PMspaces, contraction is one of the main tools to prove the existence and uniqueness of a fixed point. In 1996, a group of mathematicians Chang, Lee, Cho, Chen, Kang and Jung presented a research paper in which they obtained a generalized contraction mapping principle in PM-spaces and applied it to prove the existence theorems of solutions to differential equations in these spaces. In 1968, the concept of fuzzy sets was introduced by Zadeh. Various authors, for example, Deng, Ereeg, Fang , Kaleva and Seikkala , Kramosil and Michalek have Summary introduced the concept of fuzzy metric spaces in different ways. Fixed-point theory in fuzzy metric spaces for different contractivetype mappings is closely related to that in probabilistic metric spaces (refer. Various authors, for example, Hadžić and Pap ,Razani and Shirdaryazdi, Razani and Kouladgar and Liu and Li have studied the applications of fixed point theorems in PM-spaces to fuzzy metric spaces.

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